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LETTER TO THE EDITOR

A note on vertex models and knot polynomials

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Abstract. Vertex weights of the two-dimensional lattice integrable models with a spin-$\frac{1}{2}$ representation of SU(2), which give us a formula to calculate knot polynomials, are considered. Such weights, given previously, have one subtlety; namely, they yield an extra minus sign under some topological change of a two-dimensional projection of knots. Here we show the modified vertex weights which eliminate such a minus sign.

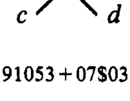
It is now well known that the Jones polynomial of knot theory [1] and its generalizations [2–4] are closely related to the lattice-integrable models of two-dimensional statistical mechanics [5].

The two-dimensional lattice-integrable statistical models can be classified into three types of model: spin models, vertex models and IRF (interaction around the face) models. The spin models, such as the Ising model [6], consist of spin variables on the lattice site, interacting usually through nearest-neighbour couplings. In the vertex models [7, 8], statistical variables lie on bonds connecting neighbouring lattice sites and interactions are assigned on each lattice site at which typically four bonds meet. The IRF models [8] have the statistical variables on two plaquettes (or dual lattice sites) with interactions among plaquettes that share a common lattice site.

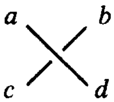
This classification, however, is not rigid. In many cases one can reformulate one kind of model in terms of another. For example, one can associate a model, which has natural formulations in terms of both the vertex and the IRF models, with every representation R of every compact Lie group G [9]. These example have also played an important role in the emergence of the concept of quantum groups [10–16].

On the other hand the Jones polynomial and its generalizations have an intimate connection with two-dimensional conformal field theory [17–21]. They can also be regarded as the vacuum expectation values of non-intersecting Wilson loops in three-dimensional topological quantum gauge field theory [5].

In the third paper of [5], the mapping from the IRF states to the vertex states is discussed. The explicit formulae for the vertex weights in the case where all statistical variables are spin-$\frac{1}{2}$ states of SU(2) are



$$= R^{ab}_{cd} = q^{1/4}\delta^a_b\delta^c_d - q^{-1/4}\varepsilon^{ab}\varepsilon_{cd}(q^{-1/4})^{\varepsilon^{ab}+\varepsilon_{cd}} \qquad (1)$$

 $= \bar{R}^{ab}_{cd} = q^{-1/4}\delta^a_c\delta^b_d - q^{1/4}\varepsilon^{ab}\varepsilon_{cd}(q^{-1/4})^{\varepsilon^{ab}+\varepsilon_{cd}}$ (2)

 $= U^{ab} = \varepsilon^{ab}(q^{-1/4})^{\varepsilon^{ab}}$ (3)

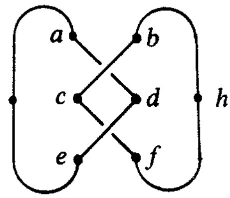
 $= U_{ab} = \varepsilon_{ab}(q^{-1/4})^{\varepsilon_{ab}}.$ (4)

Here the statistical variables placed on bonds take + or − and the convention for the SU(2) invariant antisymmetric tensors we use here is

$$\varepsilon^{+-} = \varepsilon_{+-} = 1. \tag{5}$$

By using the formulae (1)-(4), we can evaluate any vacuum expectation values (or equivalently knot polynomials) of any knots or links of non-intersecting knots. For example,

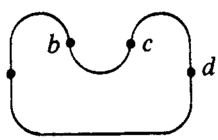
 $= \sum_{a,b}$  $= \sum_{a,b} U^{ab}U_{ab} = q^{1/2} + q^{-1/2}$ (6)

 $= \sum$ 

$$= \sum U_{ga}U^{ge}R^{ab}_{cd}R^{cd}_{ef}\,U_{bh}U^{fh}$$

$$= q^{3/2} + q^{1/2} + q^{-1/2} + q^{-3/2}. \tag{7}$$

Here we denoted the vacuum expectation value by $\langle\ \rangle$. These formulae, however, contain one difficulty. Namely

 $= \sum$ 

$$= \sum U_{ab}U^{bc}U_{cd}U^{ad} = -(q^{1/2} + q^{-1/2})$$

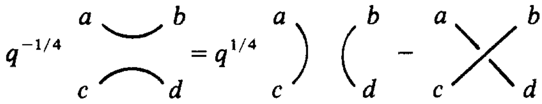
$= -$ . (8)

This minus sign comes out because of the following identity:

 $= \sum_c U^{ac}U_{cb} =$  $= \sum_c U_{bc}U^{ca} = -\delta^a_b.$ (9)

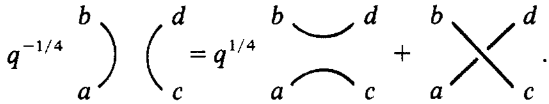
So, one must be concerned about this subtle minus sign in using the formulae (1)-(4). In the remainder of this letter, we will reconsider the origin of the formulae (1)-(4) and look for a possible modification of them which eliminates this minus sign.

In constructing vertex weight formulae, the basic ingredient is the so-called skein theory for SU(2) [5]. That is



(10)

and



(11)

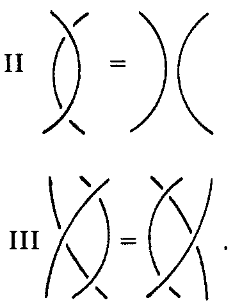
These relations can be regarded as q-deformations of the following SU(2) identity:

$$\varepsilon^{ab}\varepsilon_{cd} = \delta^a_c\delta^b_d - \delta^a_d\delta^b_c \tag{12}$$

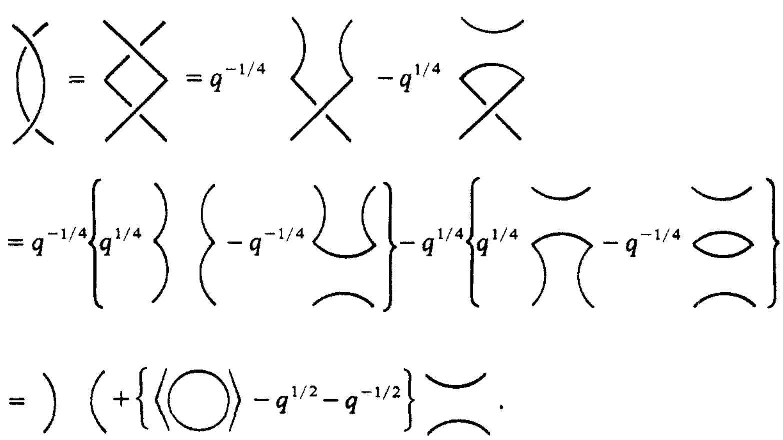
$$\delta^b_a\delta^d_c = \varepsilon^{bd}\varepsilon_{ac} + \delta^b_c\delta^d_a . \tag{13}$$

Note that because of the pseudoreality of the spin-$\frac{1}{2}$ representation of SU(2), an extra minus sign must be supplied to the second term of (11) when one considers that (11) could be derived from (10) by $\pi/2$-rotation.

Then we require invariance under the regular isotopy. This means that vacuum expectation values must be invariant under the following two Reidemeister moves,

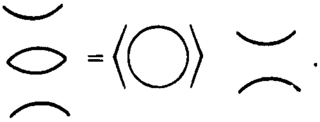


First, let us consider the Reidemeister move II. By using the skein relations (10) and (11), the left-hand side of the II is deformed in the following way:



(14)

Here we have used a factorization property of the vacuum expectation value

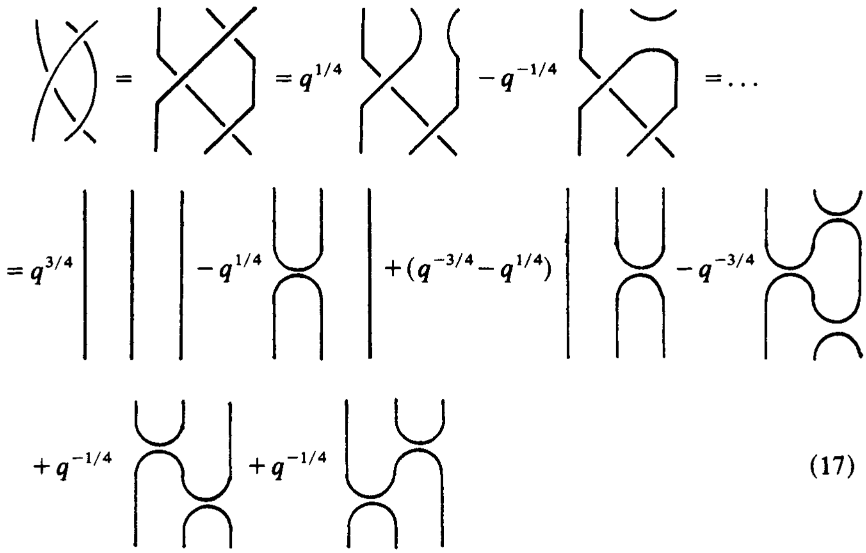


(15)

Then the invariance under the Reidemeister move II determines the vacuum expectation value of an unknotted loop as

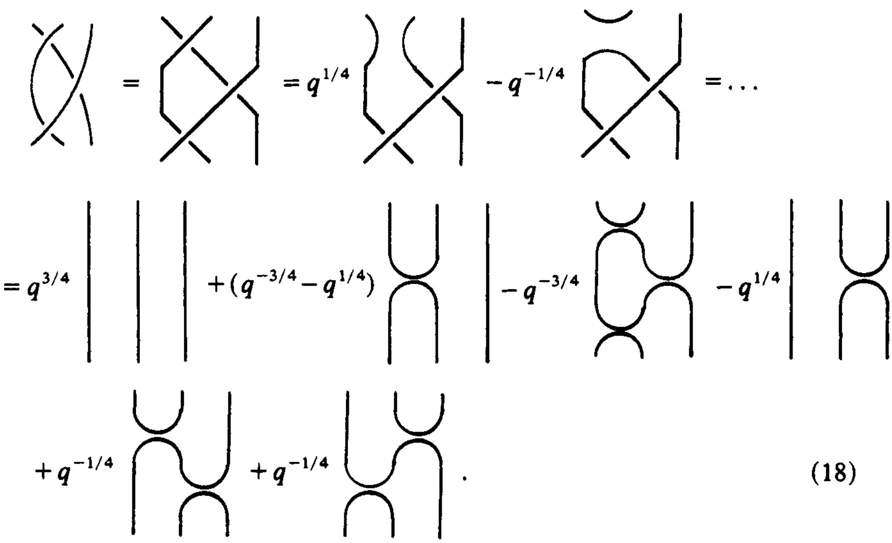
 $= q^{1/2} + q^{-1/2}.$ (16)

Second, let us deform the both sides of the Reidemeister move III. The left-hand side is



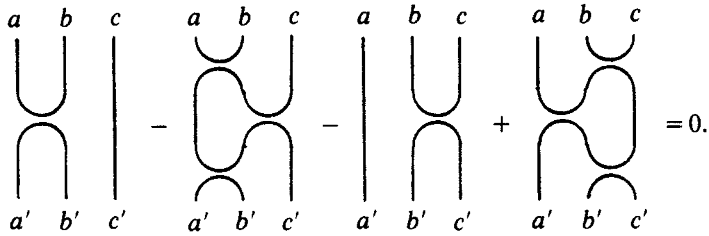
(17)

and the right-hand side is



(18)

Then from (17) and (18) we obtain

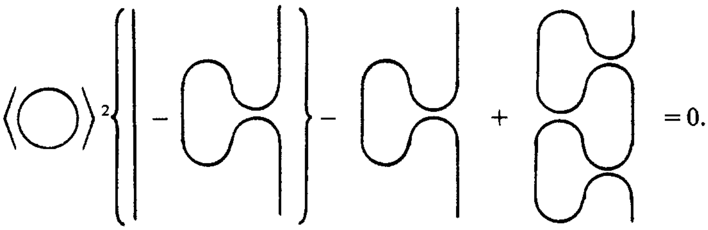


(19)

By multiplying

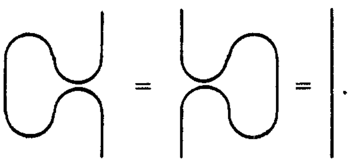
 and 

by (19), we arrive at the identity



(20)

This equation must be satisfied for any value of q, so



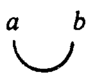
(21)

Equation (21) has the following two solutions:

(22a)

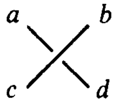
(22b)

The formulae (1)-(4) correspond to the solution (22a) but this solution has the subtlety which we explained in (8). Thus we look for the other vertex weights which correspond to the solution (22b). Because (16) is a very stringent condition there remains little possibility to deform the matrices U^{ab} and U_{ab} given in (3) and (4). Fortunately, however, the solution to (22b) can be derived from that of (22a) via a slight modification. That is

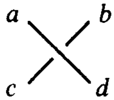
 $= \tilde{U}^{ab} = -\mathrm{i}\varepsilon^{ab}(\mathrm{i}q^{-1/4})^{\varepsilon^{ab}}$ (23)

$$= \tilde{U}_{ab} = -\mathrm{i}\varepsilon_{ab}(\mathrm{i}q^{-1/4})^{\varepsilon_{ab}}. \tag{24}$$

These matrices clearly satisfy (22*b*) and (16). Then the vertex weights are constructed through skein relations (10) and (11),

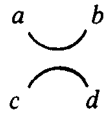


$$= q^{1/4}\delta^a_c\delta^b_d - q^{-1/4}\tilde{U}^{ab}\tilde{U}_{cd} = q^{1/4}\delta^a_c\delta^b_d + q^{-1/4}\varepsilon^{ab}\varepsilon_{cd}(\mathrm{i}q^{-1/4})^{\varepsilon^{ab}+\varepsilon_{cd}} \tag{25}$$



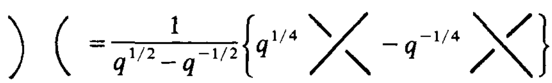
$$= q^{-1/4}\delta^a_c\delta^b_d - q^{1/4}\tilde{U}^{ab}\tilde{U}_{cd} = q^{-1/4}\delta^a_c\delta^b_d + q^{1/4}\varepsilon^{ab}\varepsilon_{cd}(\mathrm{i}q^{-1/4})^{\varepsilon^{ab}+\varepsilon_{cd}} \tag{26}$$

because

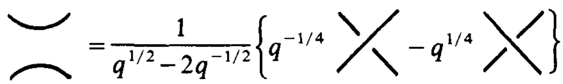


$$= \tilde{U}^{ab}\tilde{U}_{cd}. \tag{27}$$

From the skein relations (10) and (11), we have

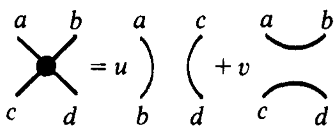


(28)



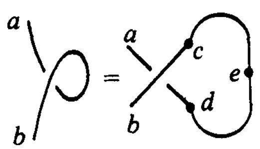
(29)

Then the 'blob' for the six-vertex model weight in the modified formulae (23)–(26) is



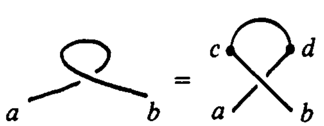
$$= u\delta^a_c\delta^b_d + v\tilde{U}^{ab}\tilde{U}_{cd}. \tag{30}$$

Although the subtle minus sign in (9) has been eliminated in the modified formulae (23)–(26), there still remains an extra minus sign contribution under $\pi/2$-rotation which comes from the pseudoreality of the spin-$\frac{1}{2}$ representation of SU(2). Namely, on the one hand,



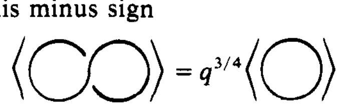
$$= \sum R^{ac}_{bd}\tilde{U}^{de}\tilde{U}_{ce} = q^{3/4}\delta^a_b \tag{31}$$

and, on the other hand,

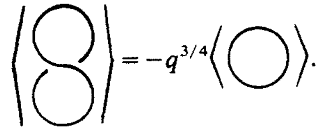


$$= \sum \bar{R}^{cd}_{ab}\tilde{U}_{cd} = -q^{3/4}\tilde{U}_{ab}. \tag{32}$$

because of this minus sign



(33)



(34)

Before closing this letter we make a comment concerning the quantum group. The pair annihilation matrix U^{ab} (or $\tilde{U}^{ab}$) and the pair creation matrix U_{ab} (or $\tilde{U}_{ab}$) can be considered as the q-deformation of the SU(2) invariant antisymmetric tensors ε^{ab} and ε_{ab} respectively. Indeed, these matrices remain invariant under the action of $\mathrm{SU}_q(2)$ (or more generally $\mathrm{SL}_q(2)$). Namely,

$$M^a_c M^b_d U^{cd} = U^{ab} \qquad (M^a_c M^b_d \tilde{U}^{cd} = \tilde{U}^{ab}) \tag{35}$$

$$M^c_a M^d_b U_{cd} = U_{cd} \qquad (M^c_a M^d_b \tilde{U}_{cd} = \tilde{U}_{cd}). \tag{36}$$

where M^a_b is a matrix element of $M \in \mathrm{SL}_q(2)$ [22].

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